

§ 5.4 Eigenvectors and Linear Transformations

We've developed the notion of eigenvectors for $n \times n$ matrices. Since an $n \times n$ matrix is a map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \text{we'd like to generalize this to}$$
$$x \longmapsto Ax$$

linear transformations $T: V \longrightarrow V$ for any vector space V .

Suppose $\dim V = n$ and let $B = \{v_1, \dots, v_n\}$ be a basis for V . Recall that if x is a vector in V , then we can consider the coordinate vector of x with respect to B ,

$[x]_B$. In other words if

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

(remember this expression for x is unique since B is a basis!)

then $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ which is a vector of \mathbb{R}^n .

with this we can relate a transformation $V \rightarrow V$ to a matrix transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Now $[x]_{\beta}$ is the input of this matrix transformation. What is the output $[T(x)]_{\beta}$

Notice if $x = c_1 v_1 + \dots + c_n v_n$ then

$$T(x) = T(c_1 v_1 + \dots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$$

Thus

$$[T(x)]_{\beta} = c_1 [T(v_1)]_{\beta} + c_2 [T(v_2)]_{\beta} + \dots + c_n [T(v_n)]_{\beta}$$

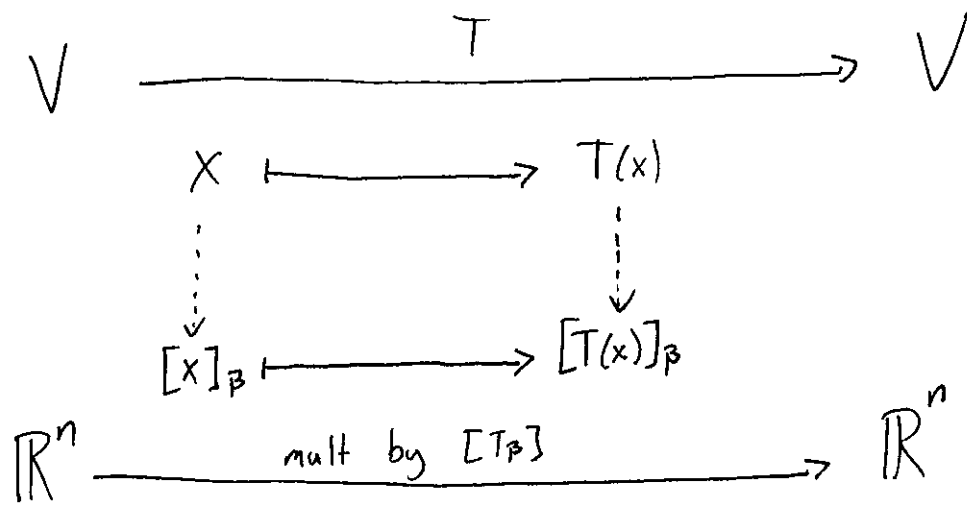
notice this can be written as a matrix multiplication:

$$[T(x)]_{\beta} = \begin{bmatrix} [T(v_1)]_{\beta} & [T(v_2)]_{\beta} & \dots & [T(v_n)]_{\beta} \end{bmatrix} \cdot [x]_{\beta}$$

Notice this is an $n \times n$ matrix and acts as a matrix transformation ~~on~~ on $[x]_{\beta}$.

We call this matrix the matrix representation of T with respect to β . We denote it by $[T]_{\beta}$.

Thus any transformation can be view as a matrix transformation.



This is useful since matrices are easy to work with.

Example

Let $T: \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be the differentiation transformation
 i.e. $T(p(t)) = p'(t)$. Find a matrix
 representation of T with respect to basis

$$B = \{1, t, t^2\}$$

$$\bullet T(1) = 0 \qquad [T(1)]_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bullet T(t) = 1 \qquad [T(t)]_{\beta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bullet T(t^2) = 2t \qquad [T(t^2)]_{\beta} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

thus

$$[T]_{\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and we can check
 this easily! \rightarrow

Let $p(t) = a + bt + ct^2$ for a, b, c real #s

$$a + bt + ct^2 \xrightarrow{T} p'(t) = b + 2ct$$

$$[a + bt + ct^2]_{\beta} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \xrightarrow{[T]_{\beta}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$$

In general lots of information can be deduced about by investigating its matrix representation (with respect to some basis).

Exercise

- Compute $\ker T$ and $\text{Null } [T]_{\beta}$. Compare them. They should convey the same information.
- Do the same with $\text{range } T$ and $\text{Col } [T]_{\beta}$

For most applications we take β to be the standard basis so its easy to compute coordinate vectors.

Defn: Let $T: V \rightarrow V$ be a linear transformation of vector space V ~~to~~ to itself. An eigenvector of T is a nonzero vector x in V with

$$T(x) = \lambda x$$

for scalar λ . We call λ the eigenvalue corresponding to eigenvector x .

Theorem

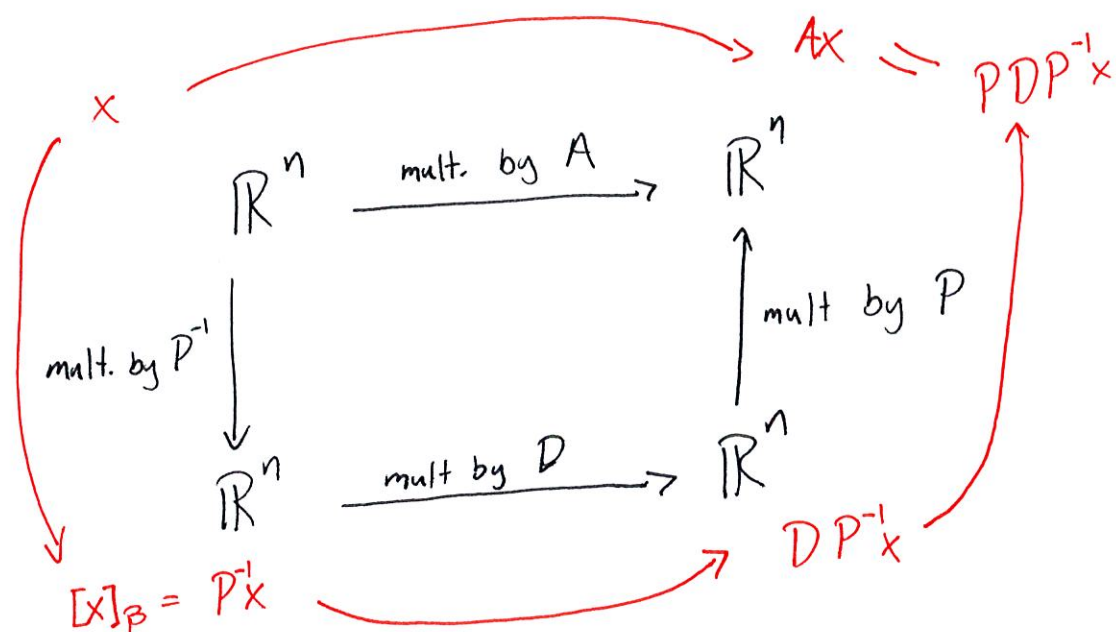
The eigenvalues of T are exactly the eigenvalues of $[T]_{\beta}$ where β is any basis of V .

Moreover, the eigenvectors of $[T]_{\beta}$ are the coordinate vectors of T .

Try this on the previous example!

Theorem (diagonal representations)

Let A be an $n \times n$ diagonalizable matrix and write $A = PDP^{-1}$. If \mathcal{B} is the basis of \mathbb{R}^n consisting of the columns of P (eigenvectors of A), then $D = [T]_{\mathcal{B}}$ where T is the matrix transformation of A , $T(x) = Ax$



Example

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the matrix transformation $T(x) = Ax$.

Find a basis of \mathbb{R}^2 such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Solution

From lesson §5.3 we found $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

so if we take $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$

$$\text{then } [T]_{\beta} = D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$