§ 5.4 Eigenvectors and Linear Transformations

We've developed the notion of eigenvectors for $n \times n$ matrices. Since and $n \times n$ matrix is a map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto Ax$$

we'd like to generalize this to linear transformations $T : V \longrightarrow V$ for any vector space V.

Suppose $\dim V = n$ and let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V. Recall that if x is a vector in V, then we can consider the <u>coordinate vector</u> of x ~~with respect to $\mathcal{B}$~~, $[x]_{\mathcal{B}}$. In other words if

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

(remember this expression for x is unique since $\mathcal{B}$ is a basis!)

then $[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ which is a vector of $\mathbb{R}^n$.

with this we can relate a transformation $V \to V$ to a matrix transformation $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Now $[x]_\beta$ is the input of this matrix transformation. What is the output $[T(x)]_\beta$

Notice if $x = c_1 v_1 + \cdots + c_n v_n$ then

$$T(x) = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

Thus

$$[T(x)]_\beta = c_1 [T(v_1)]_\beta + c_2 [T(v_2)]_\beta + \cdots + c_n [T(v_n)]_\beta$$

notice this can be written as a matrix multiplication:

$$[T(x)]_\beta = \left[[T(v_1)]_\beta \,\middle|\, [T(v_2)]_\beta \,\middle|\, \cdots \,\middle|\, [T(v_n)]_\beta \right] \cdot [x]_\beta$$

Notice this is an $n \times n$ matrix and acts as a matrix transformation on $[x]_\beta$.

We call this matrix the <u>matrix representation</u> of T with respect to B. We denote it by $[T]_\beta$.

Thus any transformation can be view as a matrix transformation.

$$
\begin{tikzcd}
V \arrow[rr, "T"] & & V \\
x \arrow[r, mapsto] \arrow[d, dashed] & T(x) \arrow[d, dashed] & \\
{[x]_\beta} \arrow[r, mapsto] & {[T(x)]_\beta} & \\
\mathbb{R}^n \arrow[rr, "\text{mult by } {[T_\beta]}"] & & \mathbb{R}^n
\end{tikzcd}
$$

This is useful since matrices are easy to work with.

__Example__

Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be the differentiation transformation i.e. $T(p(t)) = p'(t)$. Find a matrix representation of T with respect to basis $\mathcal{B} = \{1, t, t^2\}$

- $T(1) = 0$ $\qquad [T(1)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- $T(t) = 1$ $\qquad [T(t)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $T(t^2) = 2t$ $\qquad [T(t^2)]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

thus
$$[T]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and we can check this easily! $\longrightarrow$

Let $p(t) = a + bt + ct^2$ for a, b, c real #'s

$$
\begin{tikzcd}
a + bt + ct^2 \arrow[r, "T"] \arrow[d, dashed] & p'(t) = b + 2ct \arrow[d, dashed] \\
{[a + bt + ct^2]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \arrow[r, "{[T]_\beta}"] & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}
\end{tikzcd}
$$

In general lots of information can be deduced about by investigating its matrix representation (with respect to some basis).

<u>Exercise</u>

- Compute $\ker T$ and $\text{Null } [T]_\beta$. Compare them. They should convey the same information.
- Do the same with range T and $\text{Col } [T]_\beta$

For most applications we take β to be the standard basis so its easy to compute coordinate vectors.

<u>Defn</u>: Let $T : V \longrightarrow V$ be a linear transformation of vector space V ~~from~~ to itself. An <u>eigenvector</u> of T is a nonzero vector x in V with

$$T(x) = \lambda x$$

for scalar λ. We call λ the <u>eigenvalue</u> corresponding to eigenvector x.

<u>Theorem</u>
The eigenvalues of T are exactly the eigenvalues of $[T]_\beta$ where T is any basis of V.
Moreover, the eigenvectors of $[T]_\beta$ are the coordinate vectors of T.

Try this on the previous example!

Theorem (diagonal representations)

Let A be an $n \times n$ diagonalizable matrix and write $A = PDP^{-1}$. If B is the basis of $\mathbb{R}^n$ consisting of the columns of P (eigenvectors of A), then $D = [T]_B$ where T is the matrix transformation of A, $T(x) = Ax$

$$\begin{tikzcd}
x \arrow[rrr, bend left] \arrow[dddd, bend right] & & & Ax = PDP^{-1}x \\
 & \mathbb{R}^n \arrow[r, "\text{mult. by } A"] \arrow[dd, "\text{mult. by } P^{-1}"'] & \mathbb{R}^n & \\
 & & & \\
 & \mathbb{R}^n \arrow[r, "\text{mult by } D"] & \mathbb{R}^n \arrow[uu, "\text{mult by } P"'] & \\
[x]_B = P^{-1}x \arrow[rr, bend right] & & DP^{-1}x \arrow[ruuuu, bend right] &
\end{tikzcd}$$

Example

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation $T(x) = Ax$.

Find a basis of $\mathbb{R}^2$ such that $[T]_B$ is a diagonal matrix.

Solution

From lesson §5.3 we found $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

so if we take $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$

then $[T]_{\beta} = D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$